MATH 5061 Lecture 2 (Jan 20)

Problem Set 1 due next Wed. via Blackboard.

Last time: abstract manifolds / smooth maps etc......

Recall: A smooth map $f: M^m \to N^n$ is immersion / submersion at $p \in M$ iff $d_{\phi(x)}(\psi \circ f \circ \phi^{-1})$ is 1-1 / onto in any charts around $p \in M$ and $f(p) \in N$.

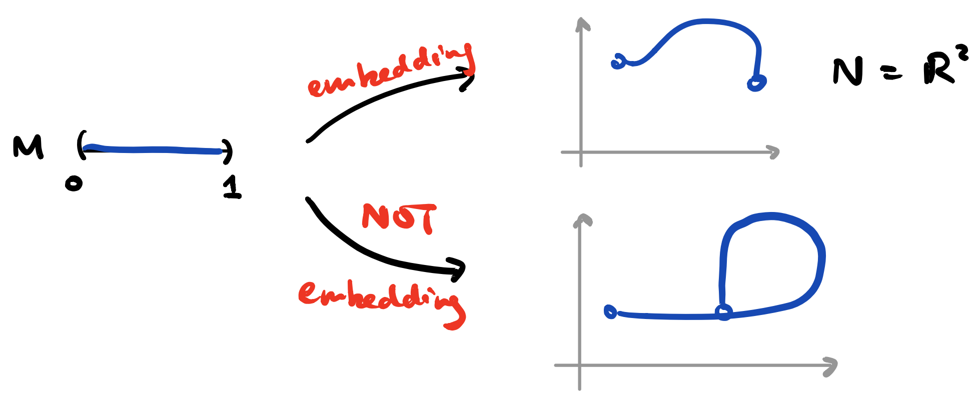
Locally (by IFT), in some local coord.,

immersion: $(m \leq n)$ $\quad f(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$

submersion: $(m \geq n)$ $\quad f(x_1, \dots, x_m) = (x_1, \dots, x_n)$

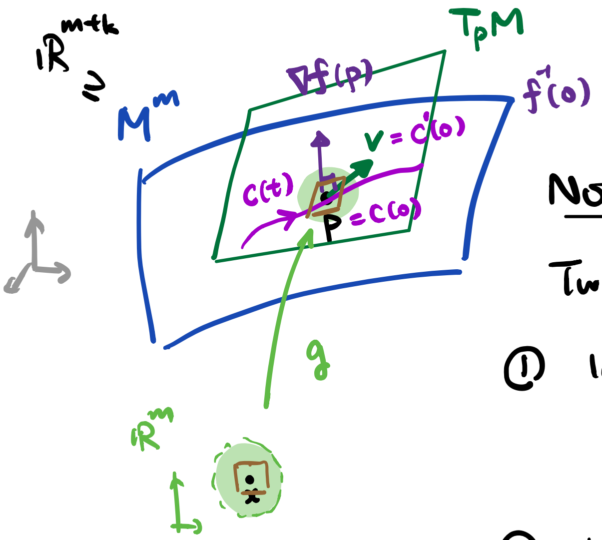
Def$^{\underline{n}}$: $f: M \to N$ embedding iff f is an immersion; and $f: M \to f(M)$ homeomorphism

Example: $M = (0,1)$, $N = \mathbb{R}^2$; $f: (0,1) \to \mathbb{R}^2$



§ Tangent Bundle

Motivation: $p \in M^m \subseteq \mathbb{R}^{m+k}$ submanifold



$$T_pM := \left\{ v \in \mathbb{R}^{m+k} \;\middle|\; \begin{array}{l} \exists \text{ smooth } c : (-\varepsilon, \varepsilon) \to M \text{ s.t.} \\ c(0) = p, \ c'(0) = v \end{array} \right\}$$

<u>Note</u>: T_pM is an m-dim'l subspace in $\mathbb{R}^{m+k}$

Two ways to describe this subspace:

① locally, $M = f^{-1}(0)$ for some $f : \mathcal{U} \subseteq \mathbb{R}^{m+k} \to \mathbb{R}^k$

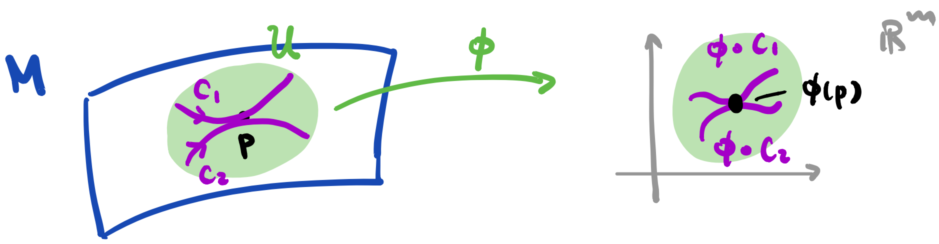
$\Rightarrow$ $T_pM = \ker(df_p)$. dim: $(m+k) - k = m$

② locally, parametrization $g : W \subseteq \mathbb{R}^m \to M \subseteq \mathbb{R}^{m+k}$

$\Rightarrow$ $T_pM = dg_x(\mathbb{R}^m)$ dim $= m$

Q: How to define T_pM in the setting of abstract manifolds?

<u>Def^n</u>: Let $p \in M$. Given curves $c_i : I_i \to M$, $i = 1, 2$, where $I_1, I_2 \subseteq \mathbb{R}$ open intervals containing 0 s.t. $c_1(0) = p = c_2(0)$.



We say $c_1 \sim c_2$ iff $\exists$ chart $(\mathcal{U}, \phi)$ around p s.t.

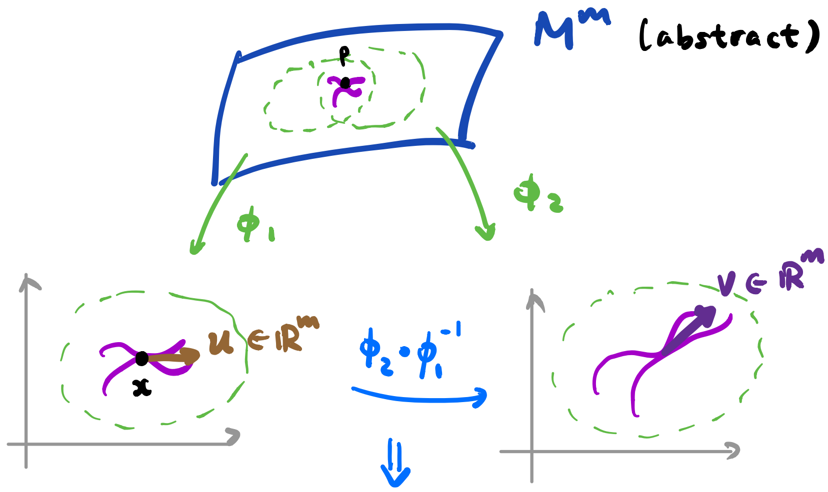
<u>Ex</u>: This is an equivalence relation.

$$(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0) \quad \text{inside } \mathbb{R}^m.$$

$$T_pM := \{ [c] \mid c : \overset{0}{\underset{\cap}{}} I \to M \text{ curve s.t. } c(0) = p \}$$

Remark: • T_pM m-dim'l (abstract) vector space.

* • The relation $\sim$ above is "chart-independent".



$$\Downarrow$$

$$v = d_x(\phi_2 \circ \phi_1^{-1})(u)$$

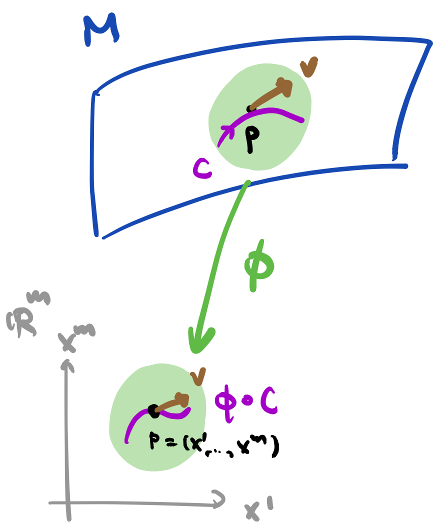
Def^{n}: $TM := \underset{p \in M}{\coprod} T_pM = \{(p,v) : p \in M,\ v \in T_pM\}.$

Tangent Bundle of M

disjoint union

Thm: TM is a smooth manifold (of $\dim = 2 \cdot \dim M$)

"**Why?**" Describe the local charts for TM.



$$(c(0), c'(0)) = (\underset{\underset{M}{\cap}}{p}, \underset{\underset{T_pM}{\cap}}{v}) \in TM$$

$$\Big\downarrow \text{local coord}$$

$$\Big(\underbrace{\phi \circ c(0)}_{\substack{(x^1,\dots,x^m) \\ \cap \\ \mathbb{R}^m}},\ \underbrace{(\phi \circ c)'(0)}_{\substack{\cap \\ \mathbb{R}^m}}\Big) \in \mathbb{R}^{2m}$$

Transition maps: $(\phi_1 \circ c(0), (\phi_1 \circ c)'(0)) \longleftrightarrow (\phi_2 \circ c(0), (\phi_2 \circ c)'(0))$

$\phi_2 \circ \phi_1^{-1}$ smooth

$d(\phi_2 \circ \phi_1^{-1})_{\phi_1(\cdot)}$ smooth

§ Vector Bundles

Def^n: A vector bundle (of rank n) consists of a map

(total space) (base space)

$$\pi : E \longrightarrow B$$

Notation:

$$\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \pi \\ B \end{array}$$

s.t. (1) E, B smooth maps, π smooth, onto

(2) $\exists$ open cover $\{U_i\}_{i \in I}$ of B and

$\exists$ diffeo $h_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$

local trivializations s.t. $h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

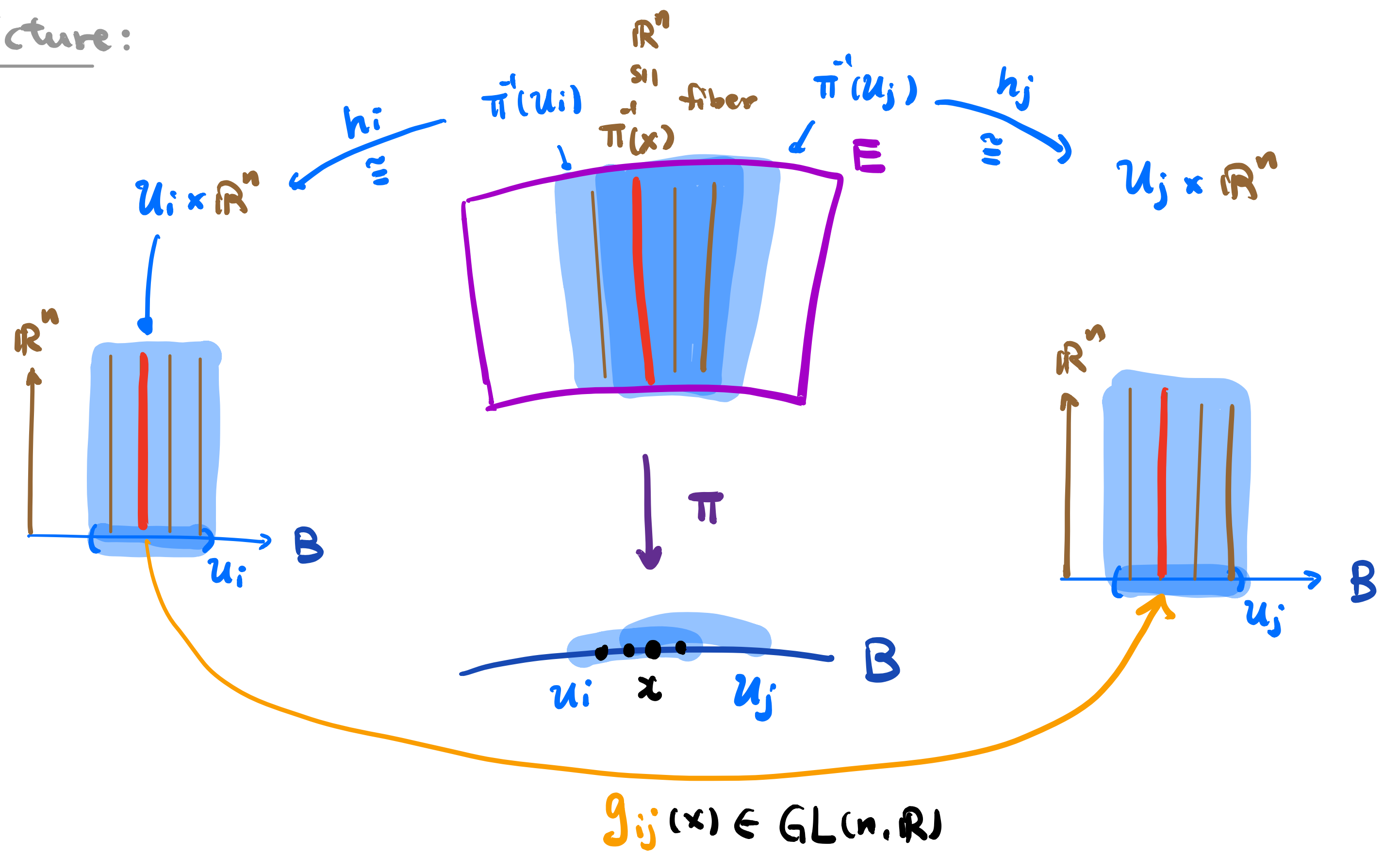
(3) The "transition maps" $h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \xrightarrow{\cong} (U_i \cap U_j) \times \mathbb{R}^n$

are diffeomorphisms of the form:

$$h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x) \cdot v)$$

where $g_{ij} : U_i \cap U_j \longrightarrow GL(n, \mathbb{R})$ smooth (in x).

Picture:



Examples: (i) $M \times \mathbb{R}^n$ "trivial bundle".

(ii) TM is a rank m vector bundle, where $m = \dim M$.

$$\begin{tikzcd} \mathbb{R}^m \arrow[r] & TM \arrow[d, "\pi"'] & \ni (p, \overset{[c]}{v}) \arrow[d, mapsto] \\ & M & \ni p \end{tikzcd}$$

$\{(\mathcal{U}_i, \phi_i)\}$ chart on M

local trivialization

$$h_i : \pi^{-1}(\mathcal{U}_i) \xrightarrow{\cong} \mathcal{U}_i \times \mathbb{R}^n$$

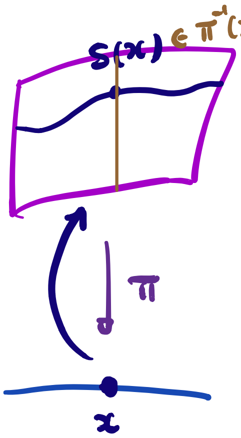
$$(p, v) \longmapsto (\phi_i(p), (\phi_i \circ c)'(0))$$

$$g_{ij}(x) = d(\phi_j \circ \phi_i^{-1})_{\phi_i(x)} \in GL(n, \mathbb{R})$$

Def$^{\underline{n}}$: A vector bundle (of rank n) $\pi : E \to B$ is trivial if $\exists$ diffeo $h : E \xrightarrow{\cong} B \times \mathbb{R}^n$ s.t. it is fiberwise linear isomorphism, i.e. $h : \pi^{-1}(x) \xrightarrow{\cong} \{x\} \times \mathbb{R}^n$.

Def$^{\underline{n}}$: A smooth map $s : B \to E$ is called a section of the vector bundle $\pi : E \to B$ if $\pi \circ s = \mathrm{id}_B$.

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \arrow[u, bend left=60, "s"] \end{tikzcd}$$



E.g.: $E = B \times \mathbb{R}^n$

A section $s : B \to \mathbb{R}^n$ is a vector-valued function

§ Vector Fields on manifolds

Let M^m be a smooth m-manifold, tangent bundle TM.

<u>Def$^{\underline{n}}$</u>: A vector field on M is just a section $X: M \to TM$ of the tangent bundle TM.

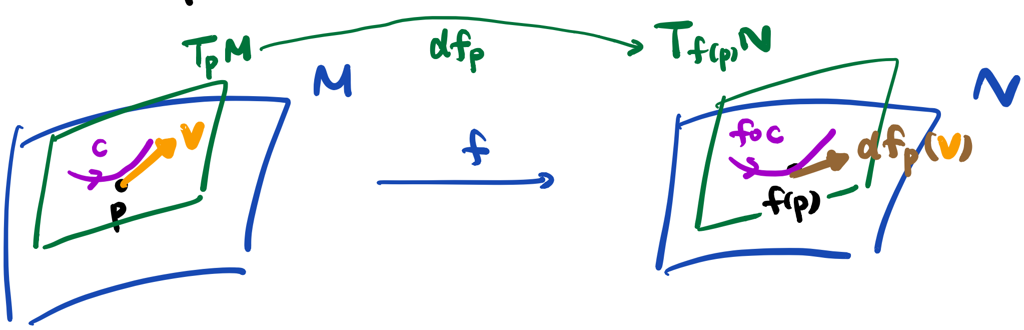
<u>Notation</u>: $\Gamma(TM) := \{\text{sections of } TM\}$ (∞-dim'l vector space)

<u>Def$^{\underline{n}}$</u>: (Pushforward of tangent vectors)

Given smooth map $f: M \to N$, and $p \in M$, then $\exists$ a linear map, differential of f at p.

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

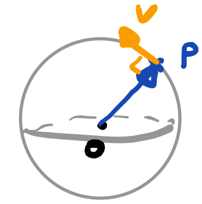
defined by $df_p(c'(0)) = (f \circ c)'(0)$ where $c: I \to M$, $c(0) = p$



<u>Note</u>: df_p is indep. of the choice of c representing $v \in T_pM$

<u>Chain Rule</u>: $\boxed{d(g \circ f)_p = dg_{f(p)} \circ df_p}$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & N \arrow[r, "g"] & P \end{tikzcd} \Rightarrow \begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[rr, bend right, "dg_{f(p)} \circ df_p"'] & T_{f(p)}N \arrow[r, "dg_{f(p)}"] & T_{g(f(p))}P \end{tikzcd}$$



Digression: Vector Fields on S^n.

$S^n \subseteq \mathbb{R}^{n+1}$ (unit sphere centered at 0)

$$TS^n = \{ (p,v) \in S^n \times \mathbb{R}^{n+1} \mid \langle p, v \rangle_{\mathbb{R}^{n+1}} = 0 \}$$

$$\Gamma(TS^n) = \{ X : S^n \to \mathbb{R}^{n+1} \text{ smooth} \mid \langle p, X(p) \rangle = 0 \;\; \forall p \in S^n \}$$

"Thm": TM trivial $\Leftrightarrow$ $\exists$ m linearly indep. vector fields on M.

Def$^{\underline{n}}$: M is parallelizable if TM is trivial.

Hard Thm 1: All closed orientable 3-manifolds are parallelizable.

Hard Thm 2: S^n is parallelizable iff $n = 1, 3$ and 7

$S^3 \times \mathbb{R}^3 \approx$

Thm: (Higher dim'l "Hairy Ball Theorem")

Any $X \in \Gamma(TS^n)$ must vanish somewhere when n is even

Remarks: • Thm $\Rightarrow$ TS^n is NOT trivial when n is even

• $n=2$ follows from Poincaré-Hopf Thm:

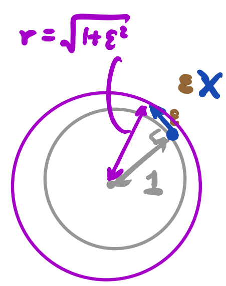
$$\sum_{\substack{p \in M \\ X(p)=0}} \text{index } X(p) = \chi(S^2) = 2 \neq 0$$

Sketch of Proof ($n \geq 4$, Milnor)

Suppose $\exists$ nowhere vanishing vector field X on S^n.

$\Downarrow$

WLOG, normalized to $\|X\| \equiv 1$.



Define $f : S^n(1) \xrightarrow{\cong} S^n(\sqrt{1+\varepsilon^2})$ diffeo $=: r$

$$x \longmapsto x + \varepsilon X(x)$$

$$d\mathrm{Vol}_{\mathbb{R}^{n+1}} = dx^0 \wedge \cdots \wedge dx^n = \frac{1}{n+1}\, d\omega$$

where $\omega := \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$ $(n-1)$ form on $\mathbb{R}^{n+1}$

polynomial in ε $\approx \int_{S^n(1)} f^*\omega \overset{\text{change of var.}}{=} \int_{S^n(r)} \omega \overset{\text{Stokes'}}{=} \int_{B^{n+1}(r)} d\omega = \underbrace{(n+1)\,\mathrm{Vol}(B^{n+1}(r))}_{(n+1)\,\mathrm{Vol}(B^{n+1}(1))\, r^{n+1}}$

$r = \sqrt{1+\varepsilon^2}$

$\| \quad c\cdot(1+\varepsilon^2)^{\frac{n+1}{2}}$

Contradiction

when n even

$\blacksquare$